MONGE'S METHOD OF INTEGRATING

$$Rr + Ss + Tt = V \longrightarrow (1)$$

where R, S, T & V are functions of x, y, z, p & q

we know that

$$p = \frac{\partial z}{\partial x} \; ; \; q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial q}{\partial x} \; ; \; t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} \quad \longrightarrow (2)$$

Now $dp = \frac{\partial p}{\partial x} \cdot dx + \frac{\partial p}{\partial y} dy = r\,dx + s\,dy \longrightarrow (3)$

$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s\,dx + t\,dy \longrightarrow (4)$

from (3) & (4)

$$r = (dp - s\,dy)/dx \quad \& \quad t = \frac{dq - s\,dx}{dy} \longrightarrow (5)$$

substituting the value of r & s given by (5) in (1) we get

$$R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V$$

or

$$(R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) + s\{R(dy)^2 - S\,dx\,dy + T(dx)^2\} = 0 \longrightarrow (6)$$

Clearly any relation between x, y, z, p & q which satisfies (6) must also satisfy the following two simultaneous equations

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \longrightarrow (7)$$

& $(dy)^2 - S\,dx\,dy + T(dx)^2 = 0 \longrightarrow (8)$

The eqn (7) & (8) are called Monge's subsidiary eqn & relation which satisfy the eqn is called intermediate integrals.

Q1: → $x^2 y'' - 2(x+1)y' + (x+2)y = (x-2)e^x$

Dividing by x, the given standard form is

$$\frac{d^2y}{dx^2} - \frac{2(x+1)}{x}\frac{dy}{dx} + \frac{x+2}{x}y = \frac{x-2}{x}e^x \quad \text{—— (1)}$$

Comparing equn (1) with $y'' + Py' + Qy = R$

we have

$$P = -\frac{(2x+2)}{x} \;;\; Q = \frac{x+2}{x} \;;\; R = \frac{x-2}{x}e^x \quad \text{—— (2)}$$

$$1 + P + Q = 1 - \frac{(2x+2)}{x} + \frac{x+2}{x} = \frac{\cancel{x} - \cancel{2x} - \cancel{2} + \cancel{x} + \cancel{2}}{x}$$

$$= 0$$

showing that $u = e^x$ —— (3) is a part of c.f. of (1).

Let the general soln of (1) be

$$y = u \cdot v \quad \longrightarrow (4)$$

Then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left(-\frac{2x+2}{x} + \frac{2}{\cancel{e^x}}\cdot\cancel{e^x}\right)\frac{dv}{dx} = \frac{x-2}{x}\cdot\frac{\cancel{e^x}}{\cancel{e^x}}$$

$$\frac{d^2v}{dx^2} + \left(-\cancel{2} - \frac{2}{x} + \cancel{2}\right)\frac{dv}{dx} = \frac{x-2}{x}$$

$$\frac{d^2v}{dx^2} - \frac{2}{x}\frac{dv}{dx} = \frac{x-2}{x}$$

Let $\frac{dv}{dx} = q$

$$\frac{dq}{dx} - \frac{2}{x}q = \frac{x-2}{x} \;;$$ which is linear in q & x.

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\int \log x} = e^{\log\frac{1}{x^2}} = \frac{1}{x^2}$$

Scanned with CamScanner

Sol^n

$$q \cdot \frac{1}{x^2} = \int \frac{x-2}{x} \cdot \frac{1}{x^2}\, dx$$

$$q \cdot \frac{1}{x^2} = \int \frac{x-2}{x^3}\, dx$$

$$= \int \left[\frac{1}{x^2} - \frac{2}{x^3}\right] dx$$

$$q \cdot \frac{1}{x^2} = \frac{-1}{x} + \frac{1}{x^2} + c_1$$

$$\because \quad q = \frac{dv}{dx}$$

$$\frac{1}{x^2} \frac{dv}{dx} = -\frac{1}{x} + \frac{1}{x^2} + c_1$$

Multiplying above eq^n by x^2

$$\frac{dv}{dx} = -x + 1 + x^2 c_1$$

$$\int dv = \int \left[-x + 1 + x^2 c_1\right] dx$$

$$v = -\frac{x^2}{2} + x + \frac{x^3}{3} c_1 + c_2 \quad \text{—— (5)}$$

Hence Sol^n is

$$y = u \cdot v$$

$$\boxed{y = e^x \left[-\frac{x^2}{2} + x + \frac{x^3}{3} c_1 + c_2\right]}$$

Ans

Q:→ Solve $(y+z)p + (z+x)q = x+y$

Soln:→ Here the Lagrange's auxillary eqn are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \longrightarrow (1)$$

Choosing 1, -1, 0 as multipliers, each fraction of (1) =

$$\frac{dx-dy}{(y+z)-(z+x)} = \frac{d(x-y)}{-(x-y)} \longrightarrow (2)$$

Again, choosing 0, 1, -1 as multipliers, each fraction of (1) =

$$= \frac{dy-dz}{(z+x)-(x+y)} = \frac{d(y-z)}{-(y-z)} \longrightarrow (3)$$

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \longrightarrow (4)$$

(2), (3) & (4)

$$\Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \longrightarrow (5)$$

Taking the first two fractions of (5),

Integrating $\log(x-y) = \log(y-z) + \log c_1$

where c_1 being an arbitrary constant.

$\log\{(x-y)/(y-z)\} = \log c_1$

or

~~$x-y = c_1(y-z)$~~ $\longrightarrow$

$$\frac{x-y}{y-z} = c_1 \longrightarrow (6)$$

Taking (6) 1^{st} & 3^{rd} fractions of (5)

$$2\frac{d(x-y)}{x-y} + \frac{d(x+y+z)}{x+y+z} = 0$$

$$2\log(x-y) + \log(x+y+z) = \log c_2$$

$$(x-y)^2(x+y+z) = c_2 \longrightarrow (7)$$

The general soln is

$$\boxed{\phi\left[\frac{x-y}{y-z},\ (x-y)^2(x+y+z)\right] = 0}$$

Ans